International Journal of Mathematics and Physical Sciences Research ISSN 2348-5736 (Online)
Vol. 10, Issue 2, pp: (40-46), Month: October 2022 - March 2023, Available at: www.researchpublish.com

Application of Differentiation under Fractional Integral Sign

Chii-Huei Yu

School of Mathematics and Statistics, Zhaoqing University, Guangdong, China

DOI: https://doi.org/10.5281/zenodo.7486538

Published Date: 27-December-2022

Abstract: **In this paper, based on Jumarie type of Riemann-Liouville (R-L) fractional calculus, we use differentiation under fractional integral sign to evaluate two improper fractional integrals. Integration by parts for fractional calculus and a new multiplication of fractional analytic functions play important roles in this article. In fact, our results are generalizations of the results in ordinary calculus.**

Keywords: **Jumarie type of R-L fractional calculus, differentiation under fractional integral sign, improper fractional integrals, integration by parts, new multiplication, fractional analytic functions.**

I. INTRODUCTION

Fractional calculus is an extension of ordinary calculus, which has a history of more than 300 years. Fractional calculus with any real or complex derivative and integral originated from Euler's work, even earlier than Leibniz's work. In recent years, fractional calculus has been widely used in many fields such as physics, biology, electrical engineering, viscoelasticity, control theory, economics [1-7].

However, the definition of fractional derivative is not unique. Common definitions include Riemann Liouville (R-L) fractional derivative, Caputo fractional derivative, Grunwald-Letnikov (G-L) fractional derivative and Jumarie's modification of R-L fractional derivative [8-12]. Since Jumarie type of R-L fractional derivative helps to avoid non-zero fractional derivative of constant function, it is easier to use this definition to connect fractional calculus with traditional calculus.

In this paper, based on the Jumarie's modified R-L fractional calculus and a new multiplication of fractional analytic functions, we study the method of differentiation under fractional integral sign. Integration by parts for fractional calculus plays an important role in this paper. In addition, we provide two examples to illustrate how to use differentiation under fractional integral sign to solve improper fractional integrals. In fact, these results we obtained are generalizations of traditional calculus results.

II. PRELIMINARIES

Firstly, we introduce the fractional calculus used in this paper.

Definition 2.1 ([13])**:** If $0 < \alpha \leq 1$, and x_0 is a real number. The Jumarie type of Riemann-Liouville (R-L) α-fractional derivative is defined by

$$\left({}_{x_0}D_x^{\alpha}\right)[f(x)] = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x_0}^{x}\frac{f(t)-f(x_0)}{(x-t)^{\alpha}}dt\ , \qquad (1)$$

And the Jumarie type of Riemann-Liouville α-fractional integral is defined by

$$\left({}_{x_0}I_x^{\alpha}\right)[f(x)] = \frac{1}{\Gamma(\alpha)}\int_{x_0}^{x}\frac{f(t)}{(x-t)^{1-\alpha}}dt\ , \qquad (2)$$

where $\Gamma(\ \ )$ is the gamma function.

In the following, some properties of Jumarie type of fractional derivative are introduced.

Proposition 2.2 ([14])**:** *If* α, β, x_0, c *are real numbers and* $\beta \geq \alpha > 0$, *then*

$$\left({}_{x_0}D_x^{\alpha}\right)\left[(x-x_0)^{\beta}\right]=\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x-x_0)^{\beta-\alpha}, \tag{3}$$

and

$$\left({}_{x_0}D_x^{\alpha}\right)[c]=0. \tag{4}$$

Next, the definition of fractional analytic function is introduced.

Definition 2.3 ([15])**:** Let x, x_0, and a_k be real numbers for all k, $x_0 \in (a,b)$, and $0<\alpha\leq 1$. If the function $f_\alpha:[a,b]\to R$ can be expressed as an α-fractional power series, that is, $f_\alpha(x^\alpha)=\sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}$ on some open interval containing x_0, then we say that $f_\alpha(x^\alpha)$ is α-fractional analytic at x_0. In addition, if $f_\alpha:[a,b]\to R$ is continuous on closed interval $[a,b]$ and it is α-fractional analytic at every point in open interval (a,b), then f_α is called an α-fractional analytic function on $[a,b]$.

In the following, we introduce a new multiplication of fractional analytic functions.

Definition 2.4 ([16])**:** If $0<\alpha\leq 1$, and x_0 is a real number. Suppose that $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are α-fractional analytic at $x=x_0$,

$$f_\alpha(x^\alpha)=\sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}, \tag{5}$$

$$g_\alpha(x^\alpha)=\sum_{k=0}^{\infty}\frac{b_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}. \tag{6}$$

Then

$$\begin{aligned}&f_\alpha(x^\alpha)\otimes g_\alpha(x^\alpha)\\&=\sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}\otimes\sum_{k=0}^{\infty}\frac{b_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}\\&=\sum_{k=0}^{\infty}\frac{1}{\Gamma(k\alpha+1)}\left(\sum_{m=0}^{k}\binom{k}{m}a_{k-m}b_m\right)(x-x_0)^{k\alpha}.\end{aligned} \tag{7}$$

Equivalently,

$$\begin{aligned}&f_\alpha(x^\alpha)\otimes g_\alpha(x^\alpha)\\&=\sum_{k=0}^{\infty}\frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k}\otimes\sum_{k=0}^{\infty}\frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k}\\&=\sum_{k=0}^{\infty}\frac{1}{k!}\left(\sum_{m=0}^{k}\binom{k}{m}a_{k-m}b_m\right)\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k}.\end{aligned} \tag{8}$$

Definition 2.5: Assume that $0<\alpha\leq 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are two α-fractional analytic functions. Then $\left(f_\alpha(x^\alpha)\right)^{\otimes n}=f_\alpha(x^\alpha)\otimes\cdots\otimes f_\alpha(x^\alpha)$ is called the n-th power of $f_\alpha(x^\alpha)$. On the other hand, if $f_\alpha(x^\alpha)\otimes g_\alpha(x^\alpha)=1$, then $g_\alpha(x^\alpha)$ is called the $\otimes$ reciprocal of $f_\alpha(x^\alpha)$, and is denoted by $\left(f_\alpha(x^\alpha)\right)^{\otimes -1}$.

Definition 2.6 ([16])**:** Assume that $0<\alpha\leq 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are α-fractional analytic at $x=x_0$,

$$f_\alpha(x^\alpha)=\sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}=\sum_{k=0}^{\infty}\frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k}, \tag{9}$$

$$g_\alpha(x^\alpha)=\sum_{k=0}^{\infty}\frac{b_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}=\sum_{k=0}^{\infty}\frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k}. \tag{10}$$

The compositions of $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are defined by

$$(f_\alpha\circ g_\alpha)(x^\alpha)=f_\alpha\left(g_\alpha(x^\alpha)\right)=\sum_{k=0}^{\infty}\frac{a_k}{k!}\left(g_\alpha(x^\alpha)\right)^{\otimes k}, \tag{11}$$

and

$$(g_\alpha \circ f_\alpha)(x^\alpha) = g_\alpha(f_\alpha(x^\alpha)) = \sum_{k=0}^{\infty} \frac{b_k}{k!} (f_\alpha(x^\alpha))^{\otimes k}. \tag{12}$$

Definition 2.7 ([17])**:** Suppose that $0 < \alpha \leq 1$, and x is a real number. The α-fractional exponential function is defined by

$$E_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{x^{k\alpha}}{\Gamma(k\alpha+1)} = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes k}. \tag{13}$$

The α-fractional logarithmic function $Ln_\alpha(x^\alpha)$ is the inverse function of $E_\alpha(x^\alpha)$**.** In addition, the α-fractional cosine and sine function are defined as follows:

$$cos_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k\alpha}}{\Gamma(2k\alpha+1)} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes 2k}, \tag{14}$$

and

$$sin_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{(2k+1)\alpha}}{\Gamma((2k+1)\alpha+1)} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes (2k+1)}. \tag{15}$$

Theorem 2.8 (integration by parts for fractional calculus) ([18])**:** *Assume that* $0 < \alpha \leq 1$, a, b *are real numbers, and* $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ *are* α*-fractional analytic functions, then*

$$\left({}_aI_b^\alpha\right)\left[f_\alpha(x^\alpha) \otimes \left({}_aD_x^\alpha\right)[g_\alpha(x^\alpha)]\right] = [f_\alpha(x^\alpha) \otimes g_\alpha(x^\alpha)]_{x=a}^{x=b} - \left({}_aI_b^\alpha\right)\left[g_\alpha(x^\alpha) \otimes \left({}_aD_x^\alpha\right)[f_\alpha(x^\alpha)]\right]. \tag{16}$$

III. RESULTS AND EXAMPLES

In the following, we introduce the method of differentiation under fractional integral sign. First, we need a lemma.

Lemma 3.1: *If* $0 < \alpha \leq 1$, t *is a nonzero real variable, and* $f_\alpha(x^\alpha)$ *is a* α*-fractional analytic function at* $x = 0$*, then*

$$\frac{d}{dt} f_\alpha(tx^\alpha) = \frac{1}{t} \cdot \frac{1}{\Gamma(\alpha+1)} x^\alpha \otimes \left({}_0D_x^\alpha\right)[f_\alpha(tx^\alpha)]. \tag{17}$$

Proof Let $f_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{k!} \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes k}$, then

$$f_\alpha(tx^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{k!} \left(t \cdot \frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes k} = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes k}. \tag{18}$$

Hence,

$$\begin{aligned}
&\frac{d}{dt} f_\alpha(tx^\alpha) \\
&= \frac{d}{dt}\left[\sum_{k=0}^{\infty} \frac{a_k}{k!} t^k \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes k}\right] \\
&= \sum_{k=0}^{\infty} \frac{a_k}{k!} \frac{d}{dt}(t^k) \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes k} \\
&= \sum_{k=1}^{\infty} \frac{a_k}{k!} k t^{k-1} \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes k} \\
&= \sum_{k=1}^{\infty} \frac{a_k}{(k-1)!} t^{k-1} \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes k}.
\end{aligned} \tag{19}$$

On the other hand,

$$\begin{aligned}
&\frac{1}{t} \cdot \frac{1}{\Gamma(\alpha+1)} x^\alpha \otimes \left({}_0D_x^\alpha\right)[f_\alpha(tx^\alpha)] \\
&= \frac{1}{t} \cdot \frac{1}{\Gamma(\alpha+1)} x^\alpha \otimes \left({}_0D_x^\alpha\right)\left[\sum_{k=0}^{\infty} \frac{a_k}{k!} t^k \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes k}\right] \\
&= \frac{1}{t} \cdot \frac{1}{\Gamma(\alpha+1)} x^\alpha \otimes \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k \left({}_0D_x^\alpha\right)\left[\left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes k}\right] \\
&= \frac{1}{t} \cdot \frac{1}{\Gamma(\alpha+1)} x^\alpha \otimes \sum_{k=1}^{\infty} \frac{a_k}{k!} k t^k \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes (k-1)}
\end{aligned}$$

$$= \sum_{k=1}^{\infty} \frac{a_k}{(k-1)!} t^{k-1} \left(\frac{1}{\Gamma(\alpha+1)} x^{\alpha}\right)^{\otimes k} . \tag{20}$$

Therefore, the desired result holds. Q.e.d.

Theorem 3.2 (differentiation under fractional integral sign)**:** *Suppose that* $0 < \alpha \le 1$, t *is a nonzero real variable, and* $f_\alpha(x^\alpha)$ *is a* α*-fractional analytic function at* $x = 0$, *then*

$$\frac{d}{dt}\left({}_0I_x^{\alpha}\right)[f_\alpha(tx^\alpha)] = \left({}_0I_x^{\alpha}\right)\left[\frac{d}{dt} f_\alpha(tx^\alpha)\right] = \frac{1}{t}\left({}_0I_x^{\alpha}\right)\left[\frac{1}{\Gamma(\alpha+1)} x^\alpha \otimes \left({}_0D_x^{\alpha}\right)[f_\alpha(tx^\alpha)]\right]. \tag{21}$$

Proof Let $f_\alpha(tx^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{k!}\left(t \cdot \frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes k} = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes k}$, then

$$\frac{d}{dt}\left({}_0I_x^{\alpha}\right)[f_\alpha(tx^\alpha)]$$

$$= \frac{d}{dt}\left({}_0I_x^{\alpha}\right)\left[\sum_{k=0}^{\infty} \frac{a_k}{k!} t^k \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes k}\right]$$

$$= \frac{d}{dt}\sum_{k=0}^{\infty} \frac{a_k}{k!} t^k \left({}_0I_x^{\alpha}\right)\left[\left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes k}\right]$$

$$= \frac{d}{dt}\left[\sum_{k=0}^{\infty} \frac{a_k}{(k+1)!} t^k \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes (k+1)}\right]$$

$$= \sum_{k=0}^{\infty} \frac{a_k}{(k+1)!} \frac{d}{dt}\left(t^k\right) \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes (k+1)}$$

$$= \sum_{k=1}^{\infty} \frac{a_k}{(k+1)!} k t^{k-1} \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes (k+1)} . \tag{22}$$

Moreover,

$$\left({}_0I_x^{\alpha}\right)\left[\frac{d}{dt} f_\alpha(tx^\alpha)\right]$$

$$= \left({}_0I_x^{\alpha}\right)\left[\frac{d}{dt}\left[\sum_{k=0}^{\infty} \frac{a_k}{k!} t^k \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes k}\right]\right]$$

$$= \left({}_0I_x^{\alpha}\right)\left[\sum_{k=0}^{\infty} \frac{a_k}{k!} \frac{d}{dt}\left(t^k\right) \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes k}\right]$$

$$= \left({}_0I_x^{\alpha}\right)\left[\sum_{k=1}^{\infty} \frac{a_k}{k!} k t^{k-1} \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes k}\right]$$

$$= \sum_{k=1}^{\infty} \frac{a_k}{k!} k t^{k-1} \left({}_0I_x^{\alpha}\right)\left[\left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes k}\right]$$

$$= \sum_{k=1}^{\infty} \frac{a_k}{(k-1)!} t^{k-1} \frac{1}{k+1} \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes (k+1)}$$

$$= \sum_{k=1}^{\infty} \frac{a_k}{(k+1)!} k t^{k-1} \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes (k+1)} . \tag{23}$$

By Lemma 3.1, the desired results hold. Q.e.d.

Next, we give two examples to illustrate how to use differentiation under fractional integral sign to evaluate improper fractional integrals. At first, two lemmas are needed.

Lemma 3.3: *Let* $0 < \alpha \le 1$, *and* t *be a real number. Then the* α*-fractional integral*

$$\left({}_0I_x^{\alpha}\right)[E_\alpha(-tx^\alpha) \otimes sin_\alpha(x^\alpha)] = -\frac{1}{t^2+1} E_\alpha(-tx^\alpha) \otimes [t sin_\alpha(x^\alpha) + cos_\alpha(x^\alpha)] + \frac{1}{t^2+1}. \tag{24}$$

Solution By integration by parts for fractional calculus,

$$\left({}_0I_x^{\alpha}\right)[E_\alpha(-tx^\alpha) \otimes sin_\alpha(x^\alpha)]$$

$$= \left({}_0I_x^{\alpha}\right)\left[sin_{\alpha}(x^{\alpha}) \otimes \left({}_0D_x^{\alpha}\right)\left[-\frac{1}{t}E_{\alpha}(-tx^{\alpha})\right]\right]$$

$$= -\frac{1}{t}E_{\alpha}(-tx^{\alpha}) \otimes sin_{\alpha}(x^{\alpha}) - \left({}_0I_x^{\alpha}\right)\left[-\frac{1}{t}E_{\alpha}(-tx^{\alpha}) \otimes \left({}_0D_x^{\alpha}\right)[sin_{\alpha}(x^{\alpha})]\right]$$

$$= -\frac{1}{t}E_{\alpha}(-tx^{\alpha}) \otimes sin_{\alpha}(x^{\alpha}) + \frac{1}{t}\left({}_0I_x^{\alpha}\right)[E_{\alpha}(-tx^{\alpha}) \otimes cos_{\alpha}(x^{\alpha})]$$

$$= -\frac{1}{t}E_{\alpha}(-tx^{\alpha}) \otimes sin_{\alpha}(x^{\alpha}) + \frac{1}{t}\left({}_0I_x^{\alpha}\right)\left[cos_{\alpha}(x^{\alpha}) \otimes \left({}_0D_x^{\alpha}\right)\left[-\frac{1}{t}E_{\alpha}(-tx^{\alpha})\right]\right]$$

$$= -\frac{1}{t}E_{\alpha}(-tx^{\alpha}) \otimes sin_{\alpha}(x^{\alpha}) + \frac{1}{t}\left[-\frac{1}{t}E_{\alpha}(-tx^{\alpha}) \otimes cos_{\alpha}(x^{\alpha}) + \frac{1}{t} - \left({}_0I_x^{\alpha}\right)\left[-\frac{1}{t}E_{\alpha}(-tx^{\alpha}) \otimes \left({}_0D_x^{\alpha}\right)[cos_{\alpha}(x^{\alpha})]\right]\right]$$

$$= -\frac{1}{t}E_{\alpha}(-tx^{\alpha}) \otimes sin_{\alpha}(x^{\alpha}) - \frac{1}{t^2}E_{\alpha}(-tx^{\alpha}) \otimes cos_{\alpha}(x^{\alpha}) + \frac{1}{t^2} - \frac{1}{t^2}\left({}_0I_x^{\alpha}\right)[E_{\alpha}(-tx^{\alpha}) \otimes sin_{\alpha}(x^{\alpha})]\,. \quad (25)$$

Therefore,

$$\left(1 + \frac{1}{t^2}\right)\left({}_0I_x^{\alpha}\right)[E_{\alpha}(-tx^{\alpha}) \otimes sin_{\alpha}(x^{\alpha})] = -\frac{1}{t}E_{\alpha}(-tx^{\alpha}) \otimes sin_{\alpha}(x^{\alpha}) - \frac{1}{t^2}E_{\alpha}(-tx^{\alpha}) \otimes cos_{\alpha}(x^{\alpha}) + \frac{1}{t^2}. \quad (26)$$

And hence,

$$\left({}_0I_x^{\alpha}\right)[E_{\alpha}(-tx^{\alpha}) \otimes sin_{\alpha}(x^{\alpha})]$$
$$= -\frac{t}{t^2+1}E_{\alpha}(-tx^{\alpha}) \otimes sin_{\alpha}(x^{\alpha}) - \frac{1}{t^2+1}E_{\alpha}(-tx^{\alpha}) \otimes cos_{\alpha}(x^{\alpha}) + \frac{1}{t^2+1}$$
$$= -\frac{1}{t^2+1}E_{\alpha}(-tx^{\alpha}) \otimes [tsin_{\alpha}(x^{\alpha}) + cos_{\alpha}(x^{\alpha})] + \frac{1}{t^2+1}\,.$$

Q.e.d.

Lemma 3.4: *If* $0 < \alpha \leq 1$, *and* $t > 0$. *Then the improper α-fractional integral*

$$\left({}_0I_{+\infty}^{\alpha}\right)[E_{\alpha}(-tx^{\alpha}) \otimes sin_{\alpha}(x^{\alpha})] = \frac{1}{t^2+1}\,. \quad (27)$$

Proof By Lemma 3.3,

$$\left({}_0I_{+\infty}^{\alpha}\right)[E_{\alpha}(-tx^{\alpha}) \otimes sin_{\alpha}(x^{\alpha})]$$
$$= -\frac{1}{t^2+1}E_{\alpha}(-tx^{\alpha}) \otimes [tsin_{\alpha}(x^{\alpha}) + cos_{\alpha}(x^{\alpha})] + \frac{1}{t^2+1}\Big|_{x=0}^{x=+\infty}$$
$$= \frac{1}{t^2+1}\,.$$

Theorem 3.5: *Suppose that* $0 < \alpha \leq 1$, *and* $t > 0$. *Then the improper α-fractional integral*

$$\left({}_0I_{+\infty}^{\alpha}\right)\left[sin_{\alpha}(x^{\alpha}) \otimes \left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes -1} \otimes E_{\alpha}(-tx^{\alpha})\right] = \frac{\pi}{2} - arctan(t)\,. \quad (28)$$

Proof Let

$$p(t) = \left({}_0I_{+\infty}^{\alpha}\right)\left[sin_{\alpha}(x^{\alpha}) \otimes \left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes -1} \otimes E_{\alpha}(-tx^{\alpha})\right]. \quad (29)$$

Using differentiation under fractional integral sign and Lemma 3.4 yields

$$\frac{d}{dt}p(t)$$
$$= \frac{d}{dt}\left({}_0I_{+\infty}^{\alpha}\right)\left[sin_{\alpha}(x^{\alpha}) \otimes \left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes -1} \otimes E_{\alpha}(-tx^{\alpha})\right]$$
$$= \left({}_0I_{+\infty}^{\alpha}\right)\left[sin_{\alpha}(x^{\alpha}) \otimes \left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes -1} \otimes \frac{d}{dt}E_{\alpha}(-tx^{\alpha})\right]$$
$$= -\left({}_0I_{+\infty}^{\alpha}\right)[sin_{\alpha}(x^{\alpha}) \otimes E_{\alpha}(-tx^{\alpha})]$$

$$= -\frac{1}{t^2+1}. \tag{30}$$

Thus,

$$p(t) = \int -\frac{1}{t^2+1}dt = -arctan(t) + C\,, \tag{31}$$

where C is a constant. Since $p(+\infty) = 0$, it follows that $C = \frac{\pi}{2}$. And hence, the desired result holds. Q.e.d.

Theorem 3.6: *If* $0 < \alpha \le 1$, *then the improper α-fractional integral*

$$\left({}_{-\infty}I^{\alpha}_{+\infty}\right)\left[sin_\alpha(x^\alpha) \otimes \left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes -1}\right] = \pi\,. \tag{32}$$

Proof By Theorem 3.5,

$$\left({}_{-\infty}I^{\alpha}_{+\infty}\right)\left[sin_\alpha(x^\alpha) \otimes \left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes -1}\right]$$

$$= 2\left({}_{0}I^{\alpha}_{+\infty}\right)\left[sin_\alpha(x^\alpha) \otimes \left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes -1}\right]$$

$$= 2 \cdot \frac{\pi}{2}$$

$$= \pi\,.$$

Q.e.d.

IV. CONCLUSION

In this paper, based on Jumarie's modified R-L fractional calculus, we use differentiation under fractional integral sign to solve two improper fractional integrals. Integration by parts for fractional calculus and a new multiplication of fractional analytic functions play important roles in this paper. In fact, our results are generalizations of traditional calculus results. In the future, we will continue to use these methods to study the problems in engineering mathematics and fractional calculus.

REFERENCES

[1] J. T. Machado, Fractional Calculus: Application in Modeling and Control, Springer New York, 2013.

[2] R. Hilfer (Ed.), Applications of Fractional Calculus in Physics, WSPC, Singapore, 2000.

[3] V. E. Tarasov, Mathematical economics: application of fractional calculus, Mathematics, vol. 8, no. 5, 660, 2020.

[4] C. -H. Yu, A new insight into fractional logistic equation, International Journal of Engineering Research and Reviews, vol. 9, no. 2, pp.13-17, 2021.

[5] R. L. Magin, Fractional calculus in bioengineering, 13th International Carpathian Control Conference, 2012.

[6] Mohd. Farman Ali, Manoj Sharma, Renu Jain, An application of fractional calculus in electrical engineering, Advanced Engineering Technology and Application, vol. 5, no. 2, pp. 41-45, 2016.

[7] R. C. Koeller, Applications of fractional calculus to the theory of viscoelasticity, Journal of Applied Mechanics, vol. 51, no. 2, 299, 1984.

[8] K. B. Oldham and J. Spanier, The Fractional Calculus, Academic Press, Inc., 1974.

[9] S. Das, Functional Fractional Calculus, 2nd ed. Springer-Verlag, 2011.

[10] K. S. Miller, B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations, John Wiley & Sons, New York, USA, 1993.

[11] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, Calif, USA, 1999.

[12] K. Diethelm, The Analysis of Fractional Differential Equations, Springer-Verlag, 2010.

[13] C. -H. Yu, Fractional derivative of arbitrary real power of fractional analytic function, International Journal of Novel Research in Engineering and Science, vol. 9, no. 1, pp. 9-13, 2022.

[14] U. Ghosh, S. Sengupta, S. Sarkar and S. Das, Analytic solution of linear fractional differential equation with Jumarie derivative in term of Mittag-Leffler function, American Journal of Mathematical Analysis, vol. 3, no. 2, pp. 32-38, 2015.

[15] C. -H. Yu, Study of fractional analytic functions and local fractional calculus, International Journal of Scientific Research in Science, Engineering and Technology, vol. 8, no. 5, pp. 39-46, 2021.

[16] C. -H. Yu, A study on arc length of nondifferentiable curves, Research Inventy: International Journal of Engineering and Science, vol. 12, no. 4, pp. 18-23, 2022.

[17] C. -H. Yu, Research on fractional exponential function and logarithmic function, International Journal of Novel Research in Interdisciplinary Studies, vol. 9, no. 2, pp. 7-12, 2022.

[18] C. -H. Yu, Differential properties of fractional functions, International Journal of Novel Research in Interdisciplinary Studies, vol. 7, no. 5, pp. 1-14, 2020.